UNIVERSAL OPERATOR ALGEBRAS OF DIRECTED GRAPHS

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ABSTRACT. Given a directed graph, there exists a universal operator algebra and universal C^*-algebra associated to the directed graph. For finite graphs this algebra decomposes as the universal free product of some building block operator algebras. For countable directed graphs, the universal operator algebras arise as direct limits of operator algebras of finite subgraphs. Finally, a method for computing the K-groups for universal operator algebras of directed graphs is given.

In [11] Muhly associates a non-selfadjoint operator algebra to a directed graph (or quiver). Henceforth we refer to these algebras as Toeplitz quiver algebras. Kribs and Power [8] showed that the graph was a complete unitary invariant for these algebras. Recent work on these Toeplitz quiver algebras by Katsoulis and Kribs, [7] and by Solel [15], has demonstrated that the graph is a complete isomorphism invariant for these algebras. In addition Kribs and Power, [8] and [9] study the structure of these algebras, including a free product result for certain amalgamations of graphs.

In another direction [3], and [6] have initiated a study of universal operator algebras (both nonself-adjoint and self-adjoint) associated to combinatorial objects (e.g. groups, monoids, and semigroups). We have continued this study by introducing the universal operator algebra, and the universal C^*-algebra associated to a directed graph. In what follows we look at the construction of these objects, and using [6], [8], and [16] as models, we find nice decompositions. As a consequence of the decompositions we calculate the K-Theory of these algebras.

As a result of theorem 2.1 we are able to write any finite graph as a free product of copies of three "building block graphs". Using universal properties, we then show that the universal operator algebra will be a

free product of three building block algebras. For the universal C^*-algebra of a directed graph we use a construction of Blecher [2] to also decompose the universal C^*-algebra of a directed graph as free products.

For infinite graphs we are able, as in [16], to write the universal operator algebra as a direct limit of the universal operator algebras corresponding to finite subgraphs directed by inclusion. We then prove that the maximal C^*-algebra of an operator algebra preserves direct limits, and hence the same result for universal C^*-algebras of directed graphs follows.

These decompositions, with a result of Cuntz [4], allow us to compute the K-groups for the universal operator algebras of directed graphs. We are then able to extend our calculations to the K-groups of the universal C^*-algebra of a directed graph. Our calculations of the K-groups indicates that they are a poor invariant in studying these non-selfadjoint operator algebras. In fact, for a directed graph the K-groups depend only on the number of vertices.

Looking at our results on a categorical level we have associated two functors on the category of directed graphs (with directed graph homomorphisms): the first functor is into the category of operator algebras (with completely contractive homomorphisms); the second functor is into the category of C^*-algebras (with C^* homomorphisms). We show that the two functors behave well with respect to free products and direct limits. We notice that the functors presented here differ from those in [16], since the morphisms he considers are directed graph inclusions.

Before proceeding, we would like to emphasize a difference between the operator algebras in the present paper and those defined in [11]. When we construct the universal operator algebra of a directed graph we consider representations which send vertices to projections. *We do not assume that the projections are orthogonal*, as was implicit in [11]. This difference provides examples which differ significantly from the Toeplitz quiver algebras. It is also important to distinguish here between the universal free products used herein, and the "spatial" free products in [8]. Kribs and Power construct a specific representation of the free product when proving a decomposition result for directed graphs. Our decomposition, on the other hand, takes into account all representations of the two graphs, as in [3]. This yields a significant distinction, even in the case of the directed graph given by two vertices with no edges.

1. Preliminaries

In [3], Blecher and Paulsen introduced the maximal C^*-algebra of an operator algebra. For A an operator algebra recall, as in [2], the construction of the maximal C^*-algebra, $C_m^*(A)$.

Theorem 1.1 (Blecher [2] Theorem 2.1). *For a unital operator algebra A there exists a unital C^*-algebra, $C_m^*(A)$ and a completely isometric inclusion $\iota : A \to C_m^*(A)$ such that:*

(1) *$\iota(A)$ generates $C_m^*(A)$ as a C^*-algebra.*
(2) *(Universal Property) for any unital completely contractive homomorphism $\varphi : A \to C$ into a C^*-algebra C there is a unique $*$-homomorphism $\tilde{\varphi} : C_m^*(A) \to C$ making the following diagram commute*

$$
\begin{array}{ccc}
C_m^*(A) & & \\
\iota \uparrow & \searrow^{\tilde{\varphi}} & \\
A & \xrightarrow{\varphi} & C
\end{array}
$$

Remark 1.1. If A is nonunital we denote the unitization of A by A^+. By the universal property of the unitization, [10, Theorem 3.2], $C_m^*(A)$ is the C^* subalgebra of $C_m^*(A^+)$ generated by A.

Notice that by the universal property (2) of Theorem 1.1, $C_m^*(A)$ is unique up to $*$-isomorphism.

Before the construction, we recall some facts about operator algebras that can be found in [13]. For an operator algebra A, the associated operator algebra A^* is unique up to completely isometric isomorphism, independent of the particular representation of A. Further notice that for any completely contractive homomorphism $\varphi : A \to C$ into a C^*-algebra there is, by Arveson's extension formula [1, Theorem 1.29], a unique completely contractive map $\overline{\varphi} : A + A^* \to C$, which extends φ and such that $\overline{\varphi}(a^*) = \varphi(a)^*$ for $a \in A$. We will denote $\overline{\varphi}|_{A^*}$ by φ^*.

We now go through the construction of the maximal C^*-algebra as it is important in what follows.

Construction 1.1. [2, Theorem 2.1] First form the algebraic free product $A *_a A^*$ of A and A^* amalgamated over the diagonal $A \cap A^*$. This is a $*$-algebra, where $(a_1 * a_2 * \cdots * a_n)^* = (a_n^* * a_{n-1}^* * \cdots * a_1^*)$ and the involution is extended using linearity. By the universal property for $A *_a A^*$, any completely contractive representation $\varphi : A \to B(H_\varphi)$ extends to a $*$-representation, $\varphi * \varphi^*$ of $A *_a A^*$. Now define

$$\|x\| = \sup\{\|\varphi * \varphi^*(x)\|_{B(H_\varphi)} : \varphi \text{ is a completely}$$

$$\text{contractive representation of } A\}$$

for all $x \in A *_a A^*$. Completing $A *_a A^*$ with respect to this norm will give a C^*-algebra satisfying the universal properties given in Theorem 1.1 (2).

Remark 1.2. The construction shows that

$$P = \text{linspan} \cup_{n \in \mathbb{N}} \{a_1 * a_2 * \cdots * a_n : a_i \in A \cup A^*\}$$

forms a dense set in $C_m^*(A)$.

2. THE UNIVERSAL OPERATOR ALGEBRA OF A DIRECTED GRAPH

Now we let Q be a directed graph with vertex set $V(Q)$ and edge set $E(Q)$. To each edge e we denote by $s(e)$ the source vertex for E and by $r(e)$ the range vertex for e. For $n \geq 1$ we let $E(Q)^n$ be the set of words over the edges of length n, and we let $E(Q)^0$ be the set of words over the vertex set. *We allow arbitrary words, with no restriction to admissible words, as in* [11]. We define

$$w(Q) = \bigcup_{n=0}^{\infty} E(Q)^n.$$

We denote, by $|w|$, the length of the word $w \in w(Q)$ (i.e $|w| = n$ if $w \in E(Q)^n$).

Definition 2.1. For an operator algebra A we say that $\pi : Q \to A$ is a contractive representation of Q if

(1) $\pi(v)$ is a projection for all $v \in V(Q)$.
(2) $\|\pi(e)\| \leq 1$ for all $e \in E(Q)$.
(3) $\pi(v)\pi(e) = \pi(e)$ if $r(e) = v$ and $\pi(e)\pi(v) = \pi(e)$ if $s(e) = v$.

Proposition 2.1. *For a directed graph Q there exists a unique operator algebra $OA(Q)$ and a contractive representation $\iota : Q \to OA(Q)$ such that:*

(1) *$\iota(Q)$ generates $OA(Q)$ as an operator algebra.*
(2) *(Universal Property) When A is an operator algebra and $\varphi : Q \to A$ is a contractive representation there is a unique completely contractive homomorphism $\tilde{\varphi} : OA(Q) \to A$ making the following diagram commute*

$$
\begin{array}{ccc}
OA(Q) & & \\
\uparrow^{\iota} & \searrow^{\tilde{\varphi}} & \\
Q & \xrightarrow{\varphi} & A
\end{array}
$$

Remark 2.1. Notice that $OA(Q)$ is unique up to completely isometric isomorphism.

Proof. Let $\mathbb{C}w(Q)$ be the set of complex valued functions on $w(Q)$ with finite support. Define a multiplication on $\mathbb{C}w(Q)$ by

$$f * g(w) = \sum_{w_1 w_2 = w} f(w_1)g(w_2).$$

Then $\mathbb{C}w(Q)$ is an algebra with pointwise addition and multiplication as defined.

Any contractive representation $\pi : Q \to A$ extends uniquely to a representation $\tilde{\pi} : \mathbb{C}w(Q) \to A$, by sending

$$f(w) \mapsto f(w)\pi(e_1)\pi(e_2) \cdots \pi(e_n)$$

where $w = e_1 e_2 \cdots e_n$.

We define

$$\|f\|_{OA} = \sup\{\|\tilde{\pi}(f)\| : \pi \text{ is a contractive representation of } Q\}.$$

Notice that $\|\tilde{\pi}(f)\|_A \leq \|f\|_1 < \infty$ for a contractive representation π, and hence $\|f\|_{OA} < \infty$.

To verify that this is a norm we need to show that for $f \neq 0 \in \mathbb{C}w(Q)$ there exists a representation $\pi : Q \to A$ such that $\tilde{\pi}(f) \neq 0$. Let $w \in w(Q)$ be chosen so that

$$|w| = \min\{|x| : x \in \text{supp}(f)\}.$$

If $|w| = 0$, then $w = v_1 v_2 \cdots v_k$ is a product of k vertices for some k. Let $\mathbb{C}_1 * \mathbb{C}_2 * \cdots * \mathbb{C}_k$ be the nonamalgamated free product of k copies of $\mathbb{C}$ and define $\pi : Q \to \mathbb{C}_1 * \mathbb{C}_2 * \cdots \mathbb{C}_k$ by

$$\pi(v_i) = \begin{cases} 1_i & \text{where } 1_i \text{ is the unit in the } i\text{th copy of } \mathbb{C} \\ 0 & \text{else.} \end{cases}$$

Then $\tilde{\pi}(f) \neq 0$.

If $|w| > 0$, let $n = |w|$ with $w = e_1 e_2 \cdots e_n$ and consider the left regular representation of A_n, the noncommmutative analytic Toeplitz algebra, (see [5] or [14]), with generators $T_1, T_2, \cdots, T_n$. Now define $\pi : Q \to A_n$ via

$$\pi(x) = \begin{cases} 1 & x \in V(Q) \\ T_i & x = e_i \\ 0 & \text{else.} \end{cases}$$

Notice first that π is a contractive representation of Q. Secondly notice that, by the definition of the left regular representation, see the discussion in section 6 of [11], we have a faithful representation of A_n, and hence $\tilde{\pi}(f) \neq 0$. It follows that $\| \cdot \|_{OA}$ is indeed a norm.

Completing $\mathbb{C}w(Q)$ with respect to this norm yields an operator algebra satisfying the requisite universal property. $\square$

Remark 2.2. Notice that we are not claiming that A_n is universal in the sense of Theorem 2.1. In fact, it will fall out of our later analysis that this is not the case. A_n is just used as a tool to verify that $\| \cdot \|_{OA}$ is a norm.

Example 2.1. Let V_1 be the finite graph given by

$$\bullet_{v_0}$$

Here the universal operator algebra is $\mathbb{C}$.

Example 2.2. Let V_2 be the finite graph given by

$$\bullet_{v_0} \qquad \bullet_{v_1}$$

Somewhat surprisingly $OA(V_2)$ is not isomorphic to $\mathbb{C} \oplus \mathbb{C}$. In fact, this algebra turns out to be the unamalgamated universal free product $\mathbb{C} * \mathbb{C}$.

Example 2.3. We next look at the graph B_1 given by:

$$\overset{e_1}{\underset{\bullet_{v_0}}{\circlearrowleft}}$$

We claim that $OA(B_1)$ is completely isometrically isomorphic to the disk algebra, $A(\mathbb{D})$. First define the contractive representation $\iota : B_1 \to A(\mathbb{D})$ by sending v_0 to the identity and e_1 to the coordinate function $f(z) = z$. Notice also that $\iota(B_1)$ generates $A(\mathbb{D})$.

Now if $\pi : B_1 \to A$ is a contractive representation of B_1, then define the contractive homomorphism $\tilde{\pi} : A(\mathbb{D}) \to A$, via $z \mapsto \pi(e_1)$ and $1 \mapsto \pi(v_0)$. By definition this yields a contractive representation of $A(\mathbb{D})$ extending π. Now as a contractive representation of $A(\mathbb{D})$ is completely contractive [12, Corollary 3.14] the result is established.

Example 2.4. Let L_1 be the finite graph given by

$$\bullet_{v_0} \xrightarrow{e_1} \bullet_{v_1}$$
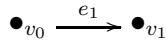
We begin by letting A_0 be the nonunital subalgebra of $A(\mathbb{D})$ generated by the coordinate function z. Consider $A = \mathbb{C} * \mathbb{C} * A_0$ and let p_0 denote the image of v_0 and p_1 denote the image of v_1, under the inclusions induced by sending one vertex to one copy of $\mathbb{C}$ and the other vertex to the other copy of $\mathbb{C}$. Letting J be the ideal in A generated by elements of the form $zp_1 - z$ and $p_2 z - z$, where z is the coordinate function in $\mathbb{C}$ then $OA(L_1) = A/J$.

Remark 2.3. These examples serve to illustrate the difference between the Toeplitz algebra of a quiver $T_+(Q)$ and the universal operator algebra $OA(Q)$. In fact, it will follow from our analysis that $T_+(Q) = OA(Q)$ only for the graphs V_0 and B_1. Notice that $T_+(V_2) = \mathbb{C} \oplus \mathbb{C}$, and $T_+(L_1)$ is the two by two upper triangular matrices.

These four examples will serve as building blocks for all finite directed graphs. We now look at how this is done. First we will need some definitions and a lemma.

Definition 2.2. We say that $k : Q \rightarrow R$ is a directed graph homomorphism if $k = (k_E, k_V)$, an ordered pair of maps, where $k_E : E(Q) \rightarrow E(R)$ and $k_V : V(E) \rightarrow V(R)$ such that $s(k_E(e)) = k_V(s(e))$ and $r(k_E(e)) = k_V(r(e))$ for all $e \in E(Q)$. We say that k is a monomorphism if k_E and k_V are both monomorphisms of sets.

Definition 2.3. We say that Q is a directed subgraph of R, denoted $Q < R$, if there exists a monomorphism $k : Q \rightarrow R$.

We will often suppress mention of the monomorphism in what follows.

Lemma 2.1. *If $Q < R$, then $OA(Q) \subseteq OA(R)$.*

Proof. Let $k : Q \rightarrow R$ be the monomorphism and denote by $\iota : R \rightarrow OA(R)$ the canonical contractive representation given in Proposition 2.1. Notice that $\iota \circ k : Q \rightarrow OA(R)$ induces a contractive representation of Q. We let $S(Q)$ denote the subalgebra of $OA(R)$ generated by $\iota \circ k(Q)$. We will show that $S(Q)$ satisfies the requisite universal property of Proposition 2.1.

Let $\pi : Q \rightarrow A$ be a contractive representation of Q. Now define $p : R \rightarrow A$ via

$$p(w) = \begin{cases} \pi(w) & \text{for } w \in k(Q) \\ 0 & \text{else.} \end{cases}$$

Notice that p is a contractive representation of R. There is then an extension $\tilde{p} : OA(R) \rightarrow A$ such that $\tilde{p} \circ \iota(w) = p(w)$ for all $w \in w(R)$. It follows that $\tilde{p}|_{S(Q)}$ is the requisite extension and hence $S(Q)$ is $OA(Q)$. $\qquad\square$

We now define the free product of a graph and relate it to the free product of operator algebras.

Definition 2.4. Let Q_1 and Q_2 be two directed graphs. Suppose R is a directed subgraph of both Q_1 and Q_2; i.e. $R < Q_1$ and $R < Q_2$. We define the vertex set of $Q_1 *_R Q_2$ by

$$V(Q_1 *_R Q_2) = (V(Q_1) \setminus V(R)) \sqcup (V(Q_2) \setminus V(R)) \sqcup V(R).$$

We let

$$E(Q_1 *_R Q_2) = (E(Q_1) \setminus E(R)) \sqcup (E(Q_2) \setminus E(R)) \sqcup E(R)$$

where $\sqcup$ is defined to be disjoint union. Notice that every $e \in E(Q_1 *_R Q_2)$ comes from a unique element of $E(Q_1)$ or $E(Q_2)$ or perhaps both. We define $r(e)$ to be the vertex in $V(Q_1 *_R Q_2)$ corresponding to the range in the original graph. We define $s(e)$ similarly.

Remark 2.4. We have not required that R be nonempty and in fact we do not want to require this. This allows us to treat disconnected graphs as unamalgamated free products of the connected components.

Remark 2.5. Any finite graph can be constructed as a finite free product of copies of V_1, V_2, L_1, and B_1. This will make the next result central to our investigations.

Recall the definition of the universal operator algebraic free product of two operator algebras A and B [3, Section 4] and its associated universal properties. Care must be exercised here, as $OA(Q)$ need not be unital. In fact we must first adjoin a unit as in [10]. It is a corollary of [10, Theorem 3.2] that the free product of A and B amalgamated over C, will be the subalgebra of $A^+ *_{C^+} B^+$ generated by A and B.

Theorem 2.1. *Let Q_1 and Q_2 be directed graphs with common directed subgraph R. Then $OA(Q_1 *_R Q_2)$ is completely isometrically isomorphic to the universal operator algebraic free product $OA(Q_1) *_{OA(R)} OA(Q_2)$.*

Proof. Notice by Lemma 2.1, that

$$OA(R) \subseteq OA(Q_1) \text{ and } OA(R) \subseteq OA(Q_2)$$

and hence $OA(Q_1) *_{OA(R)} OA(Q_2)$ is well defined. We will show that $OA(Q_1) *_{OA(R)} OA(Q_2)$ satisfies the universal properties of $OA(Q_1 *_R Q_2)$.

Notice first, for $i = 1, 2$ that there are contractive representations

$$\pi_i : Q_i \to OA(Q_1) *_{OA(R)} OA(Q_2)$$

given by $\pi_i = \tau_i \circ \iota_i$ where $\tau_i : OA(Q_i) \to OA(Q_1) *_{OA(R)} OA(Q_2)$ is the canonical inclusion, and $\iota_i : Q_i \to OA(Q_i)$ is the canonical contractive representation of Proposition 2.1. Notice that $\pi_1|_R = \pi_2|_R$. It follows that there is a contractive representation

$$\pi : Q_1 *_R Q_2 \to OA(Q_1) *_{OA(R)} OA(Q_2)$$

and that $\pi(Q_1 *_R Q_2)$ generates $OA(Q_1) *_{OA(R)} OA(Q_2)$.

Now let $\sigma : Q_1 *_R Q_2 \to A$ be a contractive representation. Then $\sigma|_{Q_1} =: \sigma_1$ and $\sigma|_{Q_2} =: \sigma_2$ are contractive representations of Q_1 and

Q_2 respectively. Furthermore, we know that $\sigma_1(h) = \sigma_2(h)$ for all words $h \in w(R)$. It follows that there exists completely contractive homomorphisms $\tilde{\sigma}_1 : OA(Q_1) \to A$ and $\tilde{\sigma}_2 : OA(Q_2) \to A$, such that $\tilde{\sigma}_1|_{OA(R)} = \tilde{\sigma}_2|_{OA(R)}$. By the universal property for free products of operator algebras there exists a completely contractive homomorphism $\tilde{\sigma} : OA(Q_1) *_{OA(R)} OA(Q_2) \to A$. The result now follows. $\qquad\square$

Example 2.5. We let Q be the following graph:
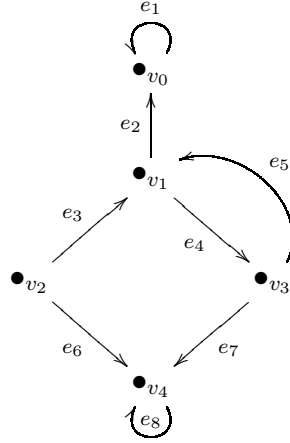
We decompose this graph into our building block graphs by first looking at the fivefold unamalgamated free product of V_0 with itself (i.e. the graph with five vertices and no edges). We will label the vertices with the labels v_0, v_1, v_2, v_3, and v_4.

$$\bullet_{v_0} \qquad \bullet_{v_1} \qquad \bullet_{v_2} \qquad \bullet_{v_3} \qquad \bullet_{v_4} \ .$$

We then amalgamate copies of L_1 with the appropriate vertices (i.e. If

$$L_1 = \quad \bullet_{u_0} \xrightarrow{\ t_1\ } \bullet_{u_1}$$

then we identify u_0 with v_1 and u_1 to v_0 to get the graph

$$\bullet_{v_0} \xleftarrow{\ t_1\ } \bullet_{v_1} \qquad \bullet_{v_2} \qquad \bullet_{v_3} \qquad \bullet_{v_4} \ ).$$

Lastly we amalgamate copies of B_1 to v_0 and v_4. This yields the decomposition

$$\left[(B_1 *_{v_0} L_1) *_{v_1} ((L_1 *_{v_2} L_1) *_{v_4} B_1) \right] *_{\{v_1, v_4\}} \left[L_1 *_{v_3} (L_1 *_{\{v_1, v_3\}} L_1) \right]$$

and hence the operator algebra can be constructed using free products.

Remark 2.6. Notice that the decomposition of the graph is not unique, but $OA(Q)$ is unique via universal considerations.

Notice that we must be careful of where the amalgamation is occurring. For example the graphs
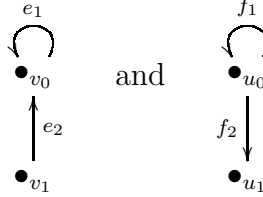
are similar but when constructing the free product the amalgamation is being taken over different copies of $\mathbb{C}$ inside $OA(L_1)$. In particular these two operator algebras are adjoints of each other, and hence anti-isomorphic. One might hope for some sort of uniqueness as in [7, Section 3] and [15, Section 3]. We are currently investigating such a result.

3. THE UNIVERSAL C^*-ALGEBRA OF A DIRECTED GRAPH

We now want to look at a construction similar to that in Section 1. Here we will be concerned with building the maximal C^*-algebra of a directed graph, as opposed to the universal operator algebra of a directed graph. This is, in spirit, very similar to a construction in Section 2 of [6]. Before we actually construct the C^*-algebra we calculate $OA(Q)^*$ for a directed graph Q.

Definition 3.1. Let Q be a directed graph. We say that $Q^{\leftarrow}$ is the adjoint of Q if $Q^{\leftarrow}$ is the graph obtained from Q by reversing all of the arrows. For notations sake, we denote by $e^{\leftarrow}$ the reversed edge associated to the edge e.

The connection with adjoints is not immediately obvious, but the next proposition justifies the name.

Proposition 3.1. *Let Q be a directed graph, then $OA(Q)^* = OA(Q^{\leftarrow})$.*

Proof. Let $\pi : OA(Q) \to B(H)$ be a completely isometric representation of $OA(Q)$. Notice that this induces a contractive representation $\pi : Q \to B(H)$. Now define a contractive representation $\pi^*(Q^{\leftarrow})$ by $\pi^*(v) = \pi(v)$ for all $v \in V(Q)$ and $\pi^*(e^{\leftarrow}) = \pi(e)^*$ for all $e \in E(Q)$. This induces a completely contractive representation $\widetilde{\pi^*} : OA(Q^{\leftarrow}) \to B(H)$ such that $\widetilde{\pi^*}(OA(Q^{\leftarrow})) = [\pi(OA(Q))]^*$. The result now follows by uniqueness of the algebra $[OA(Q)]^*$. $\qquad\square$

In section 2 we gave a method for constructing

$$OA(Q) *_{OA(V(Q))} OA(Q)^*.$$

In particular, we first construct the adjoint graph and then look at the algebra $OA(Q *_{V(Q)} Q^{\leftarrow})$. We will use a similar construction to build the maximal C^*-algebra of the graph. The universal properties will be similar to those of the maximal C^*-algebra of an operator algebra. In fact we will describe their relationship after the construction.

Definition 3.2. Let Q be a directed graph. We say that $\pi : Q *_{V(Q)} Q^{\leftarrow} \to C$ is a contractive $*$ representation of Q if π is a contractive representation such that $\pi(e)^* = \pi(e^{\leftarrow})$ for all $e \in E(Q)$.

Proposition 3.2. *For a directed graph Q there exists a C^*-algebra, $GC_m^*(Q)$ and a contractive $*$-representation $\iota : Q *_{V(Q)} Q^{\leftarrow} \to GC_m^*(Q)$ such that:*

    (1) *$\iota(Q *_{V(Q)} Q^{\leftarrow})$ generates $GC_m^*(Q)$ as a C^*-algebra.*
    (2) *(Universal Property) for any contractive $*$- representation $\varphi : Q *_{V(Q)} Q^{\leftarrow} \to C$ into a C^*-algebra C there is a unique $*$-homomorphism $\tilde{\varphi} : GC_m^*(Q) \to C$ making the following diagram commute*

$$
\begin{array}{ccc}
GC_m^*(Q) & & \\
\iota \uparrow & \diagdown^{\ \tilde{\varphi}} & \\
Q *_{V(Q)} Q^{\leftarrow} & \xrightarrow{\ \varphi\ } & C
\end{array}
$$

Remark 3.1. Notice that by the universal property, $GC_m^*(Q)$ is unique up to $*$ isomorphism.

    The construction follows a similar construction in Section 2 of [6].

Proof. We proceed as in the construction of $OA(Q)$ but this time we look at $\mathbb{C}w(Q *_{V(Q)} Q^{\leftarrow})$. We define a seminorm on $\mathbb{C}w(Q *_{V(Q)} Q^{\leftarrow})$ by setting

$$\|f\|_{C_m^*} = \sup\{\|\pi(f)\| : \pi : Q *_{V(Q)} Q^{\leftarrow} \to C \text{ is a contractive}$$
$$*\text{-representation}\}.$$

This yields a C^* seminorm on $\mathbb{C}w(Q *_{V(Q)} Q^{\leftarrow})$. Denoting by K, the kernel of the seminorm, we have a norm on $\mathbb{C}w(Q *_{V(Q)} Q^{\leftarrow})/K$. Completing with respect to this norm yields a C^*-algebra satisfying the universal properties. $\qquad\square$

Remark 3.2. Notice that $\|f\|_{C_m^*} \leq \|f\|_{OA}$ as defined previously. For those f in the image of $\iota : Q *_{V(Q)} Q^{\leftarrow} \to GC_m^*(Q)$, on the other hand, $\|f\|_{C_m^*} = \|f\|_{OA}$.

Theorem 3.1. *Let Q be a directed graph. Then $GC_m^*(Q)$ is C^* isomorphic to $C_m^*(OA(Q))$.*

Proof. The proof is just repeated applications of universal properties. Notice that by the definition of $OA(Q)$ there exists a completely contractive representation $\pi : OA(Q) \to GC^*_m(Q)$ which extends the map $\iota|_Q$. By the definition of $C^*_m(OA(Q))$ it follows that there is a $*$ homomorphism $\tilde{\pi} : C^*_m(OA(Q)) \to GC^*_m(Q)$.

There is a canonical contractive representation $\sigma : Q \to C^*_m(OA(Q))$. Extending this to a $*$ representation $\tilde{\sigma} : GC^*_m(Q) \to C^*_m(OA(Q))$ we need only verify that the two $*$ representations are inverses of each other. Notice that $\tilde{\sigma}(w) = \pi(w)$ for $w \in V(Q) \cup E(Q)$ by definition. Similarly $\tilde{\pi}(w) = \sigma(w)$ for $w \in V(Q) \cup E(Q)$. The result now follows. □

Blecher [2, Proposition 2.2] observed that for operator algebras A and B, $C^*_m(A * B) = C^*_m(A) * C^*_m(B)$ where $A * B$ is universal in the category of operator algebras with completely contractive homomorphisms and $C^*_m(A) * C^*_m(B)$ is universal in the category of C^*-algebras with $*$ homomorphisms. This result allows us to build $GC^*_m(Q)$ for finite graphs Q in a manner similar to the way we built $OA(Q)$.

Proposition 3.3. *Let Q_1 and Q_2 be finite graphs and let R be some (sub)collection of vertices in $V(Q_1) \cap V(Q_2)$. Then $GC^*_m(Q_1 *_R Q_2)$ is C^* isomorphic to*

$$GC^*_m(Q_1) *_{\mathbb{C} * \mathbb{C} \cdots * \mathbb{C}} GC^*_m(Q_2)$$

where there are $|R|$ copies of $\mathbb{C}$ in the preceding formula.

We remind the reader that we do not assume that a contractive $*$ representation of a directed graph send distinct vertices to orthogonal projections. Nor for that matter do we need assume that edges be sent to partial isometries. Notice that even in the simple case of graphs with multiple vertices and no edges we do not have the universal C^*-algebra $C^*(Q)$ of Pask, Raeburn, Renault, etc. On the other hand, notice that the universal C^*-algebra will yield a contractive $*$ representation of Q. It follows from the universal properties of $GC^*_m(Q)$ that $C^*(Q)$ is a quotient of $GC^*_m(Q)$.

4. Continuity of universal algebras with respect to direct limits

So far, we have not needed to restrict to finite graphs, although our examples have been using finite directed graphs. In this section we will give a method for dealing with infinite graphs. We begin by looking at inductive limits of operator algebras. This approach was inspired by a result of Spielberg [16, Theorem 2.35].

We say that (A_i, φ_{ij}) is an inductive system of operator algebras if each A_i is an operator algebra and each $\varphi_{ij} : A_j \rightarrow A_i$ is a completely contractive homomorphism such that $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$. Notice that if each A_i is a C^*-algebra then, since a completely contractive homomorphism is a $*$ homomorphism, this is the usual definition of an inductive system of C^*-algebras. Recall, as in [17, Appendix L], that to an inductive system (A_i, φ_{ij}) of operator algebras there exists a unique inductive limit operator algebra $A = \varinjlim A_i$ satisfying the following universal property:

> (Universal Property) There exist completely contractive homomorphisms $\Phi_i : A_i \rightarrow A$ such that $\Phi_i \circ \varphi_{ij}(x) = \Phi_j(x)$ for all $x \in A_j$. Further if B is an operator algebra such that there are completely contractive homomorphisms $\Gamma_i : A_i \rightarrow B$ with $\Gamma_i \circ \varphi_{ij}(x) = \Gamma_j(x)$ for all $x \in A_j$, then there is a unique completely contractive homomorphism $\Lambda : A \rightarrow B$ such that $\Lambda \circ \Phi_i(x) = \Gamma_i(x)$ for all $x \in A_i$ so that the following diagram commutes:

$$
\begin{array}{ccc}
A_j & \xrightarrow{\;\Phi_j\;} & A \\
{\scriptstyle\varphi_{ji}}\Big\uparrow \!\!{\scriptstyle\Gamma_j}\!\!\diagdown\!\!\diagup & & \Big| \,\Lambda \\
A_i & \xrightarrow[\;\Gamma_i\;]{\Phi_i} & B
\end{array}
$$

Theorem 4.1. *Let (A_i, φ_{ij}) be an inductive system of operator algebras. Then there exist maps $\tilde{\varphi}_{ij} : C_m^*(A_j) \rightarrow C_m^*(A_i)$ such that $(C_m^*(A_i), \tilde{\varphi}_{ij})$ is an inductive system of C^*-algebras. Furthermore*

$$\varinjlim C_m^*(A_i) = C_m^*(\varinjlim A_i).$$

Proof. By the universal property of $C_m^*(A_i)$, each $\varphi_{ij} : A_j \rightarrow A_i \subseteq C_m^*(A_i)$ will extend to a $*$ homomorphism $\tilde{\varphi}_{ij} : C_m^*(A_j) \rightarrow C_m^*(A_i)$. By uniqueness of such extensions we get that $\tilde{\varphi}_{ij} \circ \tilde{\varphi}_{jk} = \tilde{\varphi}_{ik}$. Thus $(C_m^*(A_i), \tilde{\varphi}_{ij})$ is an inductive system.

Now let $A = \varinjlim A_i$ and $C = \varinjlim C_m^*(A_i)$. We need to show that $C_m^*(A)$ satisfies the universal property unique to C. Assume that B is a C^*-algebra, and $\Gamma_i : C_m^*(A_i) \rightarrow B$ are $*$ homomorphisms. Further assume that $\sigma_i \circ \tilde{\varphi}_{ij}(x) = \sigma_j(x)$ for all $x \in C_m^*(A_j)$. Let $\overline{\sigma_j} = \sigma_j|_{A_j}$. Notice that $\overline{\sigma_j}$ is completely contractive and $\overline{\sigma_i} \circ \varphi_{ij}(x) = \overline{\sigma_j}(x)$ for all $x \in A_j$. Now by the universal property for A, there is a completely contractive homomorphism $\overline{\sigma} : A \rightarrow B$ with $\overline{\sigma} \circ \Phi_j(x) = \overline{\sigma_j}(x)$ for all $x \in A_j$. The universal property for $C_m^*(A)$ now gives a $*$ homomorphism $\sigma : C_m^*(A) \rightarrow B$ satisfying $\sigma \circ \tilde{\Phi}_j(x) = \sigma_j(x)$ for $x \in C_m^*(A_j)$. By

uniqueness of the inductive limit it follows that $C_m^*(A) \cong \varinjlim C_m^*(A_i)$.
□

Now we look at a similar result in the context of directed graphs.

Theorem 4.2. *Let Q be a directed graph and let W be any collection of subgraphs such that*

$$\cup_{F \in W} V(F) = V(Q)$$

and

$$\cup_{F \in W} E(F) = E(Q).$$

Then

$$OA(Q) = \varinjlim OA(W).$$

Proof. We show that $OA(Q)$ has the requisite universal property. First notice that by Lemma 2.1 it follows that there exists completely contractive maps $\Gamma_R : OA(R) \to OA(Q)$ for all $R \in F(Q)$. Further if $R \subseteq S \subseteq Q$ it follows that there is a connecting completely contractive homomorphism $\Gamma_{SR} : OA(R) \to OA(S)$. Now for each $R \in F(Q)$, let $\pi_R : OA(R) \to A$ be a completely contractive representation such that $\pi_S \circ \Gamma_{SR}|_{OA(R)} = \pi_R|_{OA(R)}$. Then define $\pi : Q \to A$ via $\pi(e) = \pi_E$, where E is the subgraph given by $\{e, r(e), s(e)\}$. Now by definition $\Gamma_R(OA(R)) = \Gamma_S \circ \Gamma_{SR}(OA(R))$ hence it follows that $OA(Q)$ is in fact the inductive limit. □

Corollary 4.1. *Let Q be a directed graph and W as in the previous theorem. Then $GC_m^*(Q) = \varinjlim C_m^*(Q_i)$.*

Proof. This follows directly using Theorems 4.1 and 4.2. □

We can now construct operator algebras for a graph Q with countable edge and vertex sets. We first construct the operator algebras of the finite directed subgraphs as free products of V_0, L_1 and B_1. We then use inductive limits to concretely build both $OA(Q)$ and $GC_m^*(Q)$. We will apply this technique in the next section when we discuss the K-Theory of universal operator algebras of a directed graph.

Remark 4.1. Notice that as $OA(V_2)$, $OA(L_1)$, and $A(\mathbb{D})$ are not finite dimensional it follows that $OA(Q)$ is not the direct limit of finite dimensional operator algebras. On the other hand, the result of Spielberg [16, Theorem 2.35] suggest that for the Toeplitz algebra of a quiver of Muhly [11, Section 6] continuity with respect to direct limits may yield AF algebras in certain cases.

5. K-Theory

We now use a generalized version of a result of Cuntz [4] to allow us to calculate the K-Theory using the free product decomposition of a finite directed graph.

Theorem 5.1. *Let A and B be operator algebras sharing a common C^* subalgebra D. Further assume that there exists onto idempotent completely contractive homomorphisms $\varphi_A : A \to D$ and $\varphi_B : B \to D$. Let $j_1 : D \to A$, $j_2 : D \to B$, $i_1 : A \to A *_D B$, and $i_2 : A \to A *_D B$ be the canonical inclusions.*

There is a natural split exact sequence of K-groups

$$0 \longrightarrow K_*(D) \overset{d}{\longrightarrow} K_*(A) \oplus K_*(B) \overset{i}{\longrightarrow} K_*(A *_D B) \longrightarrow 0$$

where

$$d(x) = (j_{1*}(x), -j_{2*}(x)) \quad and \quad i(x) = i_{1*}(x) + i_{2*}(x).$$

Proof. The proof of Theorem 5.1 follows in the same manner as the corresponding result in [4]. We do not give the details here. □

This allows us to compute the K-groups for free products in a rather restricted case. Fortunately, our universal graph algebras are a class of operator algebras with idempotent homomorphisms onto common subalgebras. In effect, then, this result provides a method by which we can construct the K-groups for an arbitrary directed graph with countable edge and vertex set. The actual group will have to be done on a case by case basis, but the general idea is as follows:

We first look at finite subgraphs of Q. For a finite subgraph Q' we can use the free product representation to decompose $OA(Q')$ as a finite number of free products of $OA(V_1)$, $OA(V_2)$, $OA(L_1)$, and $OA(B_1)$. Noting that the projection onto the vertices induces a completely contractive idempotent representation of $OA(Q')$ onto $OA(V(Q'))$ we can apply the above result to calculate $K_*(OA(Q'))$. Then using inductive limits we can actually calculate $K_*(OA(Q)) = \lim_{\to} K_*(OA(F(Q)))$, where $F(Q)$ is the set of finite subgraphs ordered by inclusion.

The following result allows us actual computation of the K-groups for finite graphs Q. We will also use the following result to calculate the K-groups for $GC_m^*(Q)$.

Proposition 5.1. *Let Q be a finite directed graph. Then there exists a family of completely contractive homomorphisms $\varphi_t : OA(Q) \to OA(Q)$ such that $\varphi_1 = id_{OA(Q)}$ and the range of φ_0 is $OA(V(Q))$. Moreover, $\{\varphi_t\}_{t\in[0,1]}$ is pointwise norm continuous.*

Proof. We begin by defining φ_t on a dense subalgebra of $OA(Q)$. Letting $\iota : Q \to OA(Q)$ be the canonical inclusion, define the map $\varphi_t : Q \to OA(Q)$ given by $\varphi_t(e) = t\iota(e)$ for $e \in E(Q)$ and $\varphi_t(v) = \iota(v)$ for $v \in V(Q)$. Clearly φ_t is a contractive representation for all $0 \leq t \leq 1$. Thus by the universal property there exists a completely contractive homomorphism $\tilde{\varphi}_t : OA(Q) \to OA(Q)$ extending φ_t. Clearly $\tilde{\varphi}_1 = id_{OA(Q)}$ and the range of $\tilde{\varphi}_0$ is $OA(V(Q)) \subseteq OA(Q)$.

We need only show that $\tilde{\varphi}_t$ is pointwise norm continuous. Hence let $f \in OA(Q)$, and let $\varepsilon > 0$ be given. First notice that as $\mathbb{C}(w(Q))$ is dense in $OA(Q)$ there exists some element $f_1 \neq 0 \in \mathbb{C}w(Q)$ such that $\|f_1 - f\| < \frac{\varepsilon}{3}$.

Let n be the length of the longest word in $w(Q)$ such that $f_1(w) \neq 0$. Let m be the cardinality of the set of $W = \{w \in w(Q) \setminus V(Q) : f_1(w) \neq 0\}$. Now let $M = nm\|f_1|_W\|$ and $\delta = \frac{\varepsilon}{3(M+1)}$. We claim that if $|t - t_0| < \delta$ then $\|\varphi_t(f) - \varphi_{t_0}(f)\| < \varepsilon$.

Remembering that $|w|$ is the length of a word in $w(Q)$, we notice:

$$\|\varphi_t(f) - \varphi_{t_0}(f)\|$$
$$\leq \|\varphi_t(f) - \varphi_t(f_1)\| + \|\varphi_t(f_1) - \varphi_{t_0}(f_1)\|$$
$$+ \|\varphi_{t_0}(f_1) - \varphi_{t_0}(f)\|$$
$$\leq \|f - f_1\| + \|\varphi_t(f_1) - \varphi_{t_0}(f_1)\| + \|f - f_1\|$$
$$< \frac{2\varepsilon}{3} + \|\varphi_t(f_1) - \varphi_{t_0}(f_1)\|$$
$$\leq \frac{2\varepsilon}{3} + \|\sum_{w \in W}(t^{|w|} - t_0^{|w|})f(w)\|$$
$$< \frac{2\varepsilon}{3}$$
$$+ |t - t_0| \cdot \|\sum_{w \in W}(t^{|w|-1} + t^{|w|-2}t_0 + \cdots + t_0^{|w|-1})f(w)\|$$
$$< \frac{2\varepsilon}{3} + \delta\|\sum_{w \in W} nf(w)\|$$
$$< \varepsilon.$$

The result now follows. $\square$

Hence by the homotopy invariance of K-groups

$$K_*(OA(Q)) = K_*(OA(V(Q))).$$

Thus an application of Theorem 5.1, yields

$$K_*(OA(Q)) = K_*(\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C})$$

where there are $|V(Q)|$ copies of $\mathbb{C}$ in the previous formula.

Remark 5.1. Notice that the K-groups do not constitute a very good invariant since, in effect, it merely keeps track of the vertices, while all edge data is lost.

Remark 5.2. The above calculations can be adapted for the Toeplitz algebra of a quiver and hence it can be shown that $K_*(T_+(Q)) = K_*(OA(Q))$.

We now look at some results for $C_m^*(A)$ where A is an operator algebra. These results will allow us to compute the K-groups for $GC_m^*(Q)$. To accomplish this we discuss how completely contractive endomorphisms of A extend to $*$-endomorphisms of $C_m^*(A)$ and some properties preserved by the extension. A completely contractive endomorphism $\varphi : A \to A$ can be treated as a map $\varphi : A \to C_m^*(A)$ and hence there is an extension $\tilde{\varphi} : C_m^*(A) \to C_m^*(A)$ which is a $*$ endomorphism of $C_m^*(A)$. We now start with a simple observation.

Proposition 5.2. *Let $\{\varphi_t\}_{t\in[0,1]}$ be a set of completely contractive endomorphisms of A which is point-norm continuous with respect to t. Then $\{\tilde{\varphi}_t\}_{t\in[0,1]}$ is a set of endomorphisms of $C_m^*(A)$ which is also point-norm continuous with respect to t.*

Proof. We will show that for a fixed $a \in C_m^*(A)$ we have that $\tilde{\varphi}_t(a)$ is norm continuous with respect to t. Recall that $P = A *_a A^* \subseteq C_m^*(A)$ is a dense set in $C_m^*(A)$. Notice also that continuity of φ_t^* is clear.

We first deal with elements of P of the form

(1) $$a = a_1 * a_2 * \cdots * a_m.$$

We proceed by induction on the length of the word a. If $\ell(a) = 1$ then by hypothesis $\varphi_t(a)$ is norm continuous with respect to t. Assume that $\tilde{\varphi}_t(a)$ is norm continuous for words of length $m - 1$. Letting $a = a_1 * a_2 * \cdots * a_m$ then we can rewrite a as $a_1 * b$ where $b = a_2 * a_3 \cdots * a_m$. By the induction hypothesis, given $\varepsilon > 0$ there exists $\delta > 0$ such that for $|t - t_0| < \delta$ we have $\|\tilde{\varphi}_t(a_1) - \tilde{\varphi}_{t_0}(a_1)\| < \dfrac{\varepsilon}{2(\|\tilde{\varphi}_{t_0}(b)\| + 1)}$ and

$$\|\tilde\varphi_t(b) - \tilde\varphi_{t_0}(b)\| < \frac{\varepsilon}{2(\|\tilde\varphi_t(a_1)\| + 1)}. \text{ Now for } |t - t_0| < \delta$$

$$
\begin{aligned}
\|\tilde\varphi_t(a_1 * b) - \tilde\varphi_{t_0}(a_1 * b)\| &= \|\tilde\varphi_t(a_1 * b) - \tilde\varphi_t(a_1)\tilde\varphi_{t_0}(b) \\
&\quad + \tilde\varphi_t(a_1)\tilde\varphi_{t_0}(b) - \tilde\varphi_{t_0}(a_1 * b)\| \\
&\leq \|\tilde\varphi_t(a_1 * b) - \tilde\varphi_t(a_1)\tilde\varphi_{t_0}(b)\| \\
&\quad + \|\tilde\varphi_t(a_1)\tilde\varphi_{t_0}(b) - \tilde\varphi_{t_0}(a_1 * b)\| \\
&= \|\tilde\varphi_t(a_1)\|\|\tilde\varphi_{t_0}(b) - \tilde\varphi_t(b)\| \\
&\quad + \|\tilde\varphi_t(a_1) - \tilde\varphi_{t_0}(a_1)\|\|\tilde\varphi_{t_0}(b)\| \\
&< \varepsilon.
\end{aligned}
$$

We now establish continuity for all of P. Linearity implies that $\tilde\varphi_t$ is norm continuous for $x \in P$. We can now show continuity for all of $C_m^*(A)$ since P is dense in $C_m^*(A)$. Let b be an arbitrary element of $C_m^*(A)$. For $\varepsilon > 0$ choose $b_0 \in P$ such that $\|b - b_0\| < \frac{\varepsilon}{3}$, and notice that $\|\tilde\varphi_t(b - b_0)\| < \frac{\varepsilon}{3}$ for all t as $\tilde\varphi_t$ is completely contractive for all t. Now choose $\delta > 0$ as above so that $|t - t_0| < \delta$ implies $\|\tilde\varphi_t(b_0) - \tilde\varphi_{t_0}(b_0)\|$. Now notice that

$$
\begin{aligned}
\|\tilde\varphi_t(b) - \tilde\varphi_{t_0}(b)\| &\leq \|\tilde\varphi_t(b) - \tilde\varphi_t(b_0)\| + \|\tilde\varphi_t(b_0) - \tilde\varphi_{t_0}(b_0)\| \\
&\quad + \|\tilde\varphi_{t_0}(b_0) - \tilde\varphi_{t_0}(b)\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

$\square$

As an application it follows that $K_*(OA(B_1)) = K_*(C_m^*(B_1)) = K_*(\mathbb{C})$ and also $K_*(OA(L_1)) = K_*(C_m^*(L_1)) = K_*(\mathbb{C} \oplus \mathbb{C})$. We then have the following corollaries.

Corollary 5.1. *Let Q be a finite directed graph, then $K_0(OA(Q)) = K_0(GC_m^*(Q))$ and $K_1(OA(Q)) = K_1(GC_m^*(Q))$.*

Proof. First decompose Q as a free product of copies of L_1 and B_1. As $K_*(OA(B_1)) = K_*(C_m^*(B_1))$ and $K_*(OA(L_1)) = K_*(C_m^*(L_1))$, we can use Theorem 5.1 to show that $K_*(GC_m^*(Q)) = K_*(OA(Q))$. $\square$

Corollary 5.2. *Let Q be a directed graph with countable edge and vertex sets. Then $K_*(OA(Q)) = K_*(GC_m^*(Q))$.*

Proof. This follows after applying theorem 4.2 and corollary 4.1, as K-groups are continuous with respect to direct limits. $\square$

This suggests the following question:

Question 1. For which operator algebras A is it true that $K_*(C_m^*(A)) = K_*(A)$?

From proposition 5.2, any operator algebra which has a pointwise norm continuous homotopy onto a C^* subalgebra will provide an example where the groups coincide. Hence for a counterexample one would need a non-self adjoint operator algebra which does not have a pointwise norm continuous homotopy onto the diagonal.

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